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OF MATHEMATICAL SCIENCE

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The Cambridge Colloquium, Part II

ANALYSIS SITUS

By

OSWALD VEBLEN

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THE FEBRUARY MEETING OF THE SOCIETY

The two hundred twenty-seventh regular meeting of the Society was held at Columbia University on Saturday, February 24, with the usual morning and afternoon sessions. The attendance included the following thirty-five members:

Alexander, Archibald, Barnum, W. J. Berry, Cole, Cowley, Dantzig, Fite, Fry, Gafafer, Glenn, Gronwall, Hausle, Joffe, Kasner, Langman, MacColl, Meder, F. Morley, Mullins, Northcott, Oglesby, Paaswell, Pfeiffer, Reddick, R. G. D. Richardson, Ritt, Seely, Siceloff, Sosnow, Tracey, H. E. Webb, Weisner, Whited, Whittemore.

The Secretary announced the election of the following persons to membership in the Society:

Miss Frances Almira Atwater, Milwaukee-Downer College;
Mr. Heinrich Wilhelm Brinkman, Harvard University;
Professor Robert Ernest Bruce, Boston University;
Professor Joseph Edward Burnam, Simmons College;
Professor Frederick John Dick, Theosophical University;
Mr. Lawrence Murray Graves, University of Chicago;
Mr. Albrecht Wladimir Hanush, Chicago, Ill.;
Mr. Dio Lewis Holl, University of Chicago;
Professor Anna Mayme Howe, Newcomb College;
Professor Emma Louise Konantz, Peking University;
Mr. Bernard Osgood Koopman, Harvard University;
Miss Anna Marguerite Marie Lehr, Bryn Mawr College;
Dr. Earl Emanuel Libman, University of Illinois;
Mr. Robert Bruce Lindsay, Fellow, American-Scandinavian Foundation;
Mr. Renke Gustav Lubben, University of Texas;
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Mr. Murtach Matthew Sylvester Moriarty, Holyoke, Mass.;
Mr. David Rines, Boston, Mass.;
Limmye Vernon Robinson, Meridian College;
Professor George Wellington Spenceley, Miami University;
Dr. Flora Dobler Sutton, Johns Hopkins University;
Dr. William Phillip Udinski, Texas Agricultural and Mechanical College;
Professor David Locke Webster, Stanford University;
Mr. William Marvin Whyburn, University of Texas;
Miss Constance Wiener, Smith College;
Mr. Walter Wood, Philadelphia, Pa.

Since the Annual Meeting, the following additional members of the London Mathematical Society have entered the American Mathematical Society under the reciprocity agreement:

Mr. John Brill, London;
Professor Horatio Scott Carslaw, University of Sydney;

Professor Robert William Genese, Hastings;
Mr. William Percy Dartrey MacMahon, English High School, Constantinople;
Professor John Raymond Wilton, University of Adelaide.

Twenty applications for membership in the Society were received.

The following committees were appointed:—On Arrangements for the Summer Meeting, H. S. White (chairman), G. M. Conwell, E. B. Cowley, L. D. Cummings, and the Secretary; On Policy and Budget, the President (chairman), Treasurer and Secretary, and E. T. Bell, Henry Blumberg, L. P. Eisenhart, E. R. Hedrick, and A. J. Pell; On Endowment, J. L. Coolidge (chairman), G. E. Roosevelt (secretary), Arnold Dresden, G. C. Evans, and Robert Henderson.

Professor E. V. Huntington was appointed representative of the Society in the Division of Physical Sciences of the National Research Council for the period of three years beginning July 1, 1923, as successor to Professor Veblen.

At the morning session of the Society, Ex-President Frank Morley presided, and at the afternoon session, Professor J. K. Whittemore.

Titles and abstracts of the papers read at this meeting follow below. The papers of Mr. Garabedian, Professor Hollcroft, Mr. Murray, Professor Emch, Dr. Walsh, Dr. Nelson, Professor Schwatt, Dr. Rutledge, Professor Lipka, Mr. Wilder, Mr. Rice, and Professor Kakeya were read by title. Miss Whelan's paper was read by Professor Morley. Miss Whelan was introduced by Professor Morley, Dr. Rainich by Dr. Pfeiffer, and Professor Kakeya by Professor Birkhoff.

1. Mr. C. A. Garabedian: *Circular plates of constant or variable thickness.*

This paper develops a method of series in elasticity, with particular reference to its applications to circular plates of constant or variable thickness. MM. E. and M. Cosserat, in a note in the COMPTES RENDUS, vol. 146 (1908), show how a method of series, that takes account of the geometric parameters that must enter, will yield results in agreement with classical solutions; their applications are confined to cases of

constant thickness. In his note on *Circular plates of variable thickness*, PHILOSOPHICAL MAGAZINE, May, 1922, Professor Birkhoff adopts cylindrical coordinates, with the axis of z along the axis of the plate; he then writes $z = \zeta t$, assumes the displacements developable in power series in t, and, by a method of minimizing the energy integral, obtains leading terms when the thickness is variable. The present author uses power series in t, but takes as his starting point the fundamental equations of elasticity. This results in a direct method of series, applicable to cases of variable as well as constant thickness. The method is capable of extension, and affords a solution of a problem which has interested elasticians since Poisson and Cauchy.

2. Professor J. F. Ritt: *Permutable rational functions.*

This paper investigates the circumstances under which two rational functions, $\varphi(x)$ and $\psi(x)$, are such that $\varphi[\psi(x)] = \psi[\varphi(x)]$. A detailed statement of results has appeared in the COMPTES RENDUS for January 8, 1923.

3. Professor T. R. Hollcroft: *The maximum number of cusps of a space curve.*

From a formula for the maximum genus of a space curve of given order n due to Noether, the minimum number of apparent double points is found. Consider the curve projected on a plane from a point of the nodal curve of its developable, or, for n large, from a triple point of this curve. From any point of the nodal curve, two apparent double points project into cusps and from a triple point of this curve, three. The remaining apparent double points project into nodes of the plane curve. The maximum number of cusps that may be added to the singularities of this plane curve is the number that may result from the projection of cusps of the space curve. This gives formulas for the maximum number of cusps of a space curve of given order n lying on a non-singular surface of given order μ. For $\mu = 2$ the maximum number of cusps for a given n is given by formulas involving only n, and this theorem follows: *Every proper space curve with the maximum number of cusps for a given order lies on a quadric surface.*

4. Miss A. M. Whelan: *Theory of the octavic.*

Eight numbers may be arranged into products of differences of the type $\overline{12}\cdot\overline{34}\cdot\overline{56}\cdot\overline{78}$ in 105 ways. These fall into sets of

7 such that all 28 pairs occur. Limiting these sets to those of a certain symmetrical type we find that there are 30 such sets. A sign is attached to each product and the sum of each set of 7 products formed. These sums are the Coble heptads. These 30 heptads fall into 2 sets of 15 such that an even permutation among the 8 numbers leaves each set of 15 unaltered while an odd permutation sends one set into the other. In other words, each set of 15 heptads admits the group $\frac{1}{2}$ 8! These sets are referred to as the σ and s heptads.

The following relations are readily proved: $\sum^{15}\sigma_i^n - \sum^{15}s_i^n = 0$ for $n = 1, 2, 3, 4, 5, 6, 8$; $\sum^{15}\sigma_i^7 - \sum^{15}s_i^7 = \mu\sqrt{\Delta}$, where Δ is the discriminant of the octavic whose roots are the 8 quantities, and μ is a constant which is determined by consideration of a particular case. The sums of the powers of the 30 heptads are invariants of the octavic. Invariantive conditions for a triple root are obtained.

5. Dr. G. Y. Rainich: *Analogy between electromagnetic field and analytic function.*

The counterparts of the Cauchy-Riemann equations of Volterra's generalized theory of analytic functions coincide for $n = 4$ with the electromagnetic equations for vacuum in Minkowsky form. The simplest singularity is a singular line which may be identified with an electron path. The analog of the residuum is the electric charge, and its invariance follows from the analog of Cauchy's integral theorem. There is likewise an analogy with the theory of surfaces: the energy tensor is derived from the electromagnetic tensor in the same way as the curvature from the second differential form. If we consider the curvature of the space as produced by the energy tensor, the additional members which arise in one of the sets of the Minkowsky equations by substituting absolute for ordinary differentiation make these equations non-linear in the components of the electromagnetic tensor (appreciable only near the singularities). This non-linearity may account for the equality of charges of all electrons, for the difference between positive and negative electricity, and for the dependence of the electron paths on the field.

6. Professor J. W. Alexander: *The existence of closed geodesics on surfaces.* Preliminary communication.

Associated with a surface S^2, there is a three-dimensional manifold M^3 such that a point of M^3 corresponds to a point

of S^2 together with a fixed direction through the point. The geodesics of S^2 correspond to a congruence of curves of M^3, and every closed geodesic of S^2 corresponds to a closed curve of the congruence in M^3. The author proves that the sum of the indices of the closed curves of the congruence is a topological invariant of M^3. From this may be inferred a theorem by Birkhoff on the existence of a closed geodesic on any closed surface of genus 0. The method is extended to the study of the closed geodesics of characteristic surfaces bounded by curves of zero velocity, such as arise in dynamical problems.

7. Mr. F. H. Murray: *The asymptotic expansion of the functions $W_{k,m}(z)$ of Whittaker.*

The asymptotic expansion of the functions $W_{k,m}(z)$ defined by Whittaker is already known, if z lies in a sector which does not include the negative half of the real axis; it is shown here that this expansion continues to hold near the negative half of the real axis if the parameters k, m satisfy a certain inequality. An application is made to the study of the "croissance" of the solutions of certain linear differential equations of the second order.

8. Professor Arnold Emch: *Some geometric applications of symmetric substitution groups.*

The geometry of the symmetric group in its fundamental aspects has been investigated by G. Veronese.* As the ultimate object of Veronese was a penetrating study of Pascal's hexagon inscribed in a conic, its generalizations, and of related problems, little attention was given to the various possible invariant curves and surfaces of such groups Since then the literature on problems of invariance of forms under certain groups has grown extensively.

In this paper, the author investigates a number of curves and surfaces which are invariant under the symmetric G_6 and G_{24}, associated with ternary and quaternary spaces respectively. The results obtained may perhaps be indicated by two of the theorems: *All sextics of the G_6 which pass through 5 independent sextuples of the G_6 pass through a sixth common sextuple. All space sextics of the G_{24} form a net; every sextic of the group lies on six cubic cones.*

* *Interprétation géométrique de la théorie des substitutions de n lettres particulièrement pour n = 3, 4, 5, 6, en relation avec les groupes de l'hexagramme mystique*, ANNALI DI MATEMATICA, (2), vol. 11 (1882), pp. 93–236.

9. Dr. J. L. Walsh: *A property of Haar's system of orthogonal functions.*

This note considers the problem of determining an infinite normal orthogonal system of functions $\varphi_1, \varphi_2, \varphi_3, \cdots$ so that, when $\varphi_1, \varphi_2, \cdots, \varphi_n$ are known, the function φ_{n+1} shall be chosen as that normalized function which is orthogonal to $\varphi_1, \varphi_2, \cdots, \varphi_n$ and whose total fluctuation is least. If the functions φ are required to vanish at the ends of the interval considered, the solution of the problem is precisely Haar's system.

10. Dr. C. A. Nelson: *The Riemann adjoints of certain completely integrable systems.*

Completely integrable systems of partial differential equations have been used by Wilczynski and others to investigate the projective differential properties of a surface. For a non-developable surface this system consists of two linear homogeneous equations of the second order in two independent variables. The study of developables requires a system of three equations, two of the preceding type and a third equation of the third order. The author determines all such systems whose adjoints, in the sense of Riemann, also form a completely integrable system. The surfaces whose characterizing systems of differential equations possess this property are of four types. The non-developables are (*a*) non-ruled surfaces whose invariants of lowest order vanish; (*b*) ruled surfaces which have at least one straight line directrix; (*c*) all quadrics. The developables are (*d*) quadric cones.

11. Professor Edward Kasner: *Intermediate curvatures in Riemann space.*

The Riemann curvature of an n-space at a point P depends upon the orientation of a plane or 2-flat through P. Schur proved that if the result is independent of orientation it will also be independent of P. There is then a spherical space. The Ricci curvature depends on a line, or the normal $(n-1)$-flat. If this is to be independent of orientation, the present author in an earlier paper has shown that it must also be independent of P, so the space must obey Einstein cosmological equations. He now defines a curvature depending on a k-flat, which generalizes the earlier concepts. Probably an analog of Schur's theorem is also valid here.

12. Dr. G. A. Pfeiffer: *Concerning the common boundary of two or more regions.*

The author shows that the common boundary of two or more connected regions (sets of inner points) is a continuum, i.e., a closed connected point set. Further, he shows that if the number of regions is greater than two, then the common boundary is a non-decomposable continuum (i.e., a continuum which is not the sum of two proper sub-continua) or the sum of two non-decomposable continua.

13. Dr. Louis Weisner: *Some theorems on insolvable groups.*

The object of this paper is to convert certain well known theorems concerning substitution groups into theorems on abstract groups. Among the theorems thus obtained are the following: If a group G contains a subgroup H of index n in G, such that no subgroup of G which includes H contains a subgroup invariant under G, then the order of every invariant subgroup of G is a multiple of n; if n is not a power of a prime, G is insolvable. If a group G, which is the direct product of G_1 and G_2, contains a maximal subgroup H of index n, which includes no invariant subgroup of G, and if G_1 and G_2 contain two subgroups of H which are of index n in G_1 and G_2 respectively, then G_1 and G_2 are simply isomorphic simple groups.

14. Professor I. J. Schwatt: *The value of* $\sum_{k=1}^{n} \sum_{\alpha=1}^{2k} \tan^p \alpha\pi/(2k+1)$, $\sum_{k=1}^{\infty} \prod_{\alpha=1}^{2k} \operatorname{ctn}^p \alpha\pi/(2k+1)$, *and similar forms in terms of Bernoulli and Eulerian numbers.*

The author obtains formulas connecting the preceding expressions with the Bernoulli and Eulerian numbers.

15. Professor O. E. Glenn: *On the reduction of differential parameters in terms of finite sets.* Preliminary report.

Within a functional domain of rationality defined by derivatives of the function in the arbitrary transformations on the variables, and of a certain quadratic irrationality Δ formed from the latter, the infinitude of covariantive differential parameters, involving nth order derivatives, of a binary quantic, $F = (df)^m = (d\varphi)^m$, is expressible rationally and integrally in terms of arbitrary absolute invariants together with a set of $\frac{1}{2}(n+1)(n+2)(m+1)+2$ differential parameters. This finite set is given explicitly by generalized linear invariant elements and iterated jacobian transvectants formed from such elements.

The problem of complete systems in simpler domains is under consideration.

16. Dr. George Rutledge: *A new form of Stirling's interpolation formula.*

Stirling's interpolation formula is written in a new form, and the relation to the usual form is shown. This formula is a direct consequence of a general formula for the polynomial determined by a set of evenly spaced points (JOURNAL OF MATHEMATICS AND PHYSICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, vol. 2 (1922), p. 47) and has many advantages in theoretical investigations.

17. Professor Joseph Lipka: *On the angle between two curves in V_n.*

We consider any two curves c_1 and c_2 in a general curved space of n dimensions issuing from a point P in the same direction, i.e., they have an element of arc $PQ = ds$ in common. If $d\omega$ is the angle between the curves c_1 and c_2 at the point Q, we find that $(d\omega/ds)^2 = k_1^2 + k_2^2 - 2k_1k_2 \cos\theta$, where k_1 and k_2 are the geodesic curvatures of c_1 and c_2 at P, and θ is the angle between the principal geodesic normals to c_1 and c_2 at P. If one of the curves is the geodesic through P in the direction PQ, we have immediately $d\omega/ds = k$, an intrinsic definition of the geodesic curve at a point. Compare the author's paper *Sulla curvatura geodetica delle linee appartenenti ad una varietà qualunque*, RENDICONTI, ACCADEMIA DEI LINCEI, vol. 31, May, 1922.

18. Mr. R. L. Wilder: *On accessibility of the boundary of a domain.*

In a paper presented at the April, 1922, meeting of the Society, the author showed that *in the space constituted by a plane continuous curve*, all points of the boundary of any domain are accessible if the boundary is connected im kleinen. Schoenflies has shown* that *in ordinary euclidean space of two dimensions* all points of the boundary of a plane connected domain are accessible if the boundary is a continuous curve. In the present paper it is shown that both of these results are particular cases of the following theorem: D being either a domain with respect to a continuous curve or a plane con-

* *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, JAHRESBERICHT DER VEREINIGUNG, Ergänzungsband II (1908), p. 215.

nected domain in the ordinary sense, and B its boundary, then in order that any point x of B be accessible from D it is sufficient either that (1) there exist a circle K with center at x, such that the set of all points of B that lie within K is a subset of a connected im kleinen subset $S-D$ (where S is the set of all points in the space considered) or that (2) x belong to no continuum of condensation of B.

19. Mr. L. H. Rice: *Determinantal relations based on a matrix whose elements fall into two classes.*

The author generalizes certain relations published by Muir in 1922,* one of which was based on a matrix in which the elements in a selected rectangular array were assigned to one class and the remaining elements to a second class, while another relation was based on a matrix in which the main diagonal elements were assigned to one class and the remaining elements of the matrix to a second class. Muir's proofs employed respectively Laplace's expansion and Cayley's. The present writer, by making use of a simpler mode of proof, has done away with all restrictions as to the possible distribution of the elements into two classes. The classes may be intermingled in any way, the assignment of an element to the first or second class being independent of that of any other element. Muir has also given a rule for the introduction of certain arbitrary quantities into the matrix; a second rule of the same general nature is given in the present paper.

The results are extended to p-way determinants.

20. Professor S. Kakeya: *Maximum modulus of some expressions of limited analytic functions.*

Continuing investigations previously published† the author shows that of all functions $\{f(x)\}$ which are, in the closed domain $|x| \leqq 1$, analytic and of absolute value not exceeding unity there is one and, except for a constant factor of unit absolute value, only one function $f_0(x)$ such that for $f = f_0$ the expression $\left|\sum_{r=1}^{r=m}\sum_{s=0}^{s=n_r} c_{rs} f^{(s)}(\alpha_n)\right|$, where the c_{rs} are given constants and the α_r are given constants of absolute value less than unity, assumes its least upper bound M. This function $f_0(x)$ is found to be a rational function which on the periphery of its unit circle is of constant absolute value unity.

R. G. D. Richardson,
Secretary of the Society.

*Edinburgh Royal Society Proceedings, vol. 42, p. 342.

†Tôhoku Imperial University Science Reports, vol. 6 (1917), p. 53.

RELATIONS BETWEEN KINDRED RIEMANNIAN P AND Q FUNCTIONS*

BY D. R. CURTISS

1. *Definitions and Properties of Riemannian Functions.* A Riemannian P function, as the term is used by Klein † and others, is a generalized hypergeometric function, i.e., a solution of a homogeneous linear differential equation of second order having but three singular points, all of which are regular. A Q function is a hypergeometric function which has also apparently singular points, that is, points for which the function is not singular, but which are singular points of the linear differential equation of second order satisfied by the Q function. A linear transformation carries the three regular singular points into the points 0, ∞, 1, and the differential equation of a Q function has then the form, as given in my thesis,‡

$$\frac{d^2Q}{dx^2} + p\frac{dQ}{dx} + qQ = 0, \tag{1}$$

where

$$p = \left[\frac{1-\lambda'-\lambda''}{x} + \frac{1-\nu'-\nu''}{x-1} + \sum_{i=1}^{i=k}\frac{1-\sigma_i'-\sigma_i''}{x-s_i}\right],$$

$$q = \frac{1}{x(x-1)}\left[-\frac{\lambda'\lambda''}{x} + \mu'\mu'' + \frac{\nu'\nu''}{x-1} + \sum_{i=1}^{i=k}\frac{A_i}{x-s_i}\right].$$

The sum of the constants λ, μ, ν, must be 1, 0, or a negative integer, and, as a consequence of the condition that the points s_i are not to be singular points of solutions, it follows that the σ's are zero or positive integers and the *accessory parameters* A_i verify a system of k equations each of which is quadratic or of higher degree in the A_i.

The symbol

$$Q\begin{pmatrix}\lambda', \mu', \nu', \\ \lambda'', \mu'', \nu'', x\end{pmatrix}$$

* Presented to the Society, April 14, 1922.

† See especially Klein's lithographed lectures, *Ueber die Hypergeometrische Function.* Both the terms P function and Q function were used by Riemann, *Werke*, p. 67 and p. 323.

‡ *Binary families in a triply connected region*, MEMOIRS OF THE AMERICAN ACADEMY, vol. 13 (1904), No. 1.

is used to denote any Q function which is a solution of (1). The numbers λ' and λ'' are called the *exponents* of the singular point $x = 0$, for reasons familiar in the theory of linear differential equations; μ' and μ'' are the exponents of $x = \infty$; and ν' and ν'' are the exponents of $x = 1$. If the sum of these exponents is $1 - n$, the function is said to be of order n.

For a P function the terms in $(x - s_i)$ disappear from the differential equation, and the sum of the exponents is 1. A P function is, then, a Q function of order zero, and the family of P functions satisfying a differential equation (1) free of terms in $(x - s_i)$ is completely determined by its exponents.

In Gauss's celebrated memoir on special hypergeometric functions,* it is shown that between every three such functions that are kindred, that is, whose corresponding exponents differ by integers or zero, there exists a relation linear and homogeneous in the functions, with coefficients that are polynomials in x. Riemann showed the fundamental importance of the monodromic group in these questions. Gauss's kindred functions are functions that have the same monodromic group. By the use of this conception Riemann gave a new proof of the theorem of Gauss for P functions, and was able to assign an upper limit for the degree of each coefficient.

In my thesis † I have extended this theorem to the case of three Q functions

$$Q_i\begin{pmatrix} \lambda_i', \mu_i', \nu_i', \\ \lambda_i'', \mu_i'', \nu_i'', \end{pmatrix} x, \qquad (i = 1, 2, 3), \tag{2}$$

which are *kindred* in the sense that they have the same monodromic group. In order that this be the case, corresponding exponents must be equal or differ by integers, but this condition alone is not sufficient. If, however, the Q functions belong to *irreducible* families, i.e., families of which no member is a solution of a homogeneous linear differential equation of the first order with single-valued coefficients, they will have the same group if corresponding exponents are equal or differ by integers.

* *Disquisitiones generales circa seriem infinitam*, WERKE, vol. 3, p. 123.

† Loc. cit., p. 48.

Let us write

$$\Delta_1\lambda = |(\lambda_2' - \lambda_2'') - (\lambda_3' - \lambda_3'')|,$$
$$\Delta_2\lambda = |(\lambda_3' - \lambda_3'') - (\lambda_1' - \lambda_1'')|,$$
$$\Delta_3\lambda = |(\lambda_1' - \lambda_1'') - (\lambda_2' - \lambda_2'')|,$$

and similarly for $\Delta_1\mu$, $\Delta_2\mu$, $\Delta_3\mu$, $\Delta_1\nu$, $\Delta_2\nu$, $\Delta_3\nu$. Then if the three families are irreducible and are of order n_1, n_2, n_3, respectively, three kindred functions (2) satisfy an identity

$$\phi_1 Q_1 + \phi_2 Q_2 + \phi_3 Q_3 \equiv 0, \tag{3}$$

where ϕ_1, ϕ_2, ϕ_3 are polynomials of degrees not greater than

$$\begin{aligned} &\tfrac{1}{2}(\Delta_1\lambda + \Delta_1\mu + \Delta_1\nu + n_2 + n_3 - 2), \\ &\tfrac{1}{2}(\Delta_2\lambda + \Delta_2\mu + \Delta_2\nu + n_3 + n_1 - 2), \\ &\tfrac{1}{2}(\Delta_3\lambda + \Delta_3\mu + \Delta_3\nu + n_1 + n_2 - 2), \end{aligned} \tag{4}$$

respectively. To each function Q_1 of the first family there correspond functions Q_2 and Q_3 of the other families which verify (3). From the way in which (3) is derived it follows that ϕ_1 cannot vanish identically unless there is a single-valued function F_1 such that for every function Q_2 of the second family the corresponding Q_3 of the third verifies $Q_3 = F_1 Q_2$. Similarly as regards the identical vanishing of ϕ_2 or ϕ_3.

In particular, we may take Q_2 and Q_3 as P functions, and it follows that every irreducible Q function can be expressed as a linear function with rational coefficients of two P functions. Or again, we have a similar expression for a Q function in terms of a P function and its derivative. The formulas (4) for the degrees of the coefficients ϕ in (3) enable us to supply for two well known theorems regarding relations between kindred P and Q functions proofs that have hitherto been lacking.

2. *A Linear Expression for a Q Function in Terms of a P Function and its Derivative, with Polynomial Coefficients.* In Riemann-Weber's *Die Partiellen Differentialgleichungen der Mathematischen Physik*,* there is given a brief account of some of the properties of P and Q functions. Thus we find on page 44

* Vol. 2, pp. 40–54.

$$(5)\quad x^{\delta}(1-x)^{\epsilon}Q\begin{pmatrix}\alpha, \beta, \gamma, \\ \alpha', \beta', \gamma', \end{pmatrix} x = \overline{Q}\begin{pmatrix}\alpha+\delta, \beta-\delta-\epsilon, \gamma+\epsilon, \\ \alpha'+\delta, \beta'-\delta-\epsilon, \gamma'+\epsilon, \end{pmatrix} x,$$

where δ and ϵ are any numbers we please. The function $\overline{Q}$ has the same apparently singular points with the same exponents as Q, but not, in general, the same accessory parameters. Also (p. 51), if

$$(6)\quad P = P\begin{pmatrix}\alpha, \beta, \gamma, \\ \alpha', \beta', \gamma', \end{pmatrix} x,$$

we have

$$(7)\quad \frac{dP}{dx} = Q_1\begin{pmatrix}\alpha-1, \beta+1, \gamma-1, \\ \alpha'-1, \beta'+1, \gamma'-1, \end{pmatrix} x,$$

but here we need the qualification (not mentioned in Riemann-Weber) that none of the exponents for P is zero. Combining (5) with (7), we obtain

$$(8)\quad x(1-x)\frac{dP}{dx} = \overline{Q}\begin{pmatrix}\alpha, \beta-1, \gamma, \\ \alpha', \beta'-1, \gamma', \end{pmatrix} x.$$

It is then remarked that if $A(x)$ and $B(x)$ are polynomials of degrees $n-1$ and n respectively, with no zeros at 0 or 1, the sum

$$(9)\quad A(x)x(1-x)\frac{dP}{dx} + B(x)P$$

is in general a Q function of order $2n$ with the exponent scheme

$$(10)\quad Q\begin{pmatrix}\alpha, \beta-n, \gamma, \\ \alpha', \beta'-n, \gamma', \end{pmatrix} x.$$

We may, however, choose $A(x)$ and $B(x)$ so that (9) is a Q function of order $2n-1$ whose symbol is

$$(11)\quad Q\begin{pmatrix}\alpha+1, \beta-n, \gamma, \\ \alpha', \beta'-n, \gamma', \end{pmatrix} x.$$

The question is then raised as to whether *every* Q function whose symbol is (10) or (11) can be expressed in form (9), and a proof is suggested by a count of constants in (9) and in the general Q function (10) or (11). This is unconvincing, however, because the $2n$ effective constants in (9) correspond to $2n$ possible apparently singular points in (10), for example,

and the question as to whether the accessory parameters A can be determined is unanswered.

Let us now attack this problem with the aid of (3) and (4), under the restriction that the families concerned are irreducible. In (3), let

$$Q_1 = Q, \qquad Q_2 = -\bar{Q}, \qquad Q_3 = -P,$$

as in (10), (8) and (6). Then all the numbers $\Delta\lambda$, $\Delta\mu$, $\Delta\nu$ are zero, and the degrees of ϕ_1, ϕ_2 and ϕ_3 are less than or equal to 0, $n-1$ and n, respectively. Thus we have

$$CQ = \phi_2(x)x(1-x)\frac{dP}{dx} + \phi_3(x)P, \tag{12}$$

where the degrees of ϕ_2 and ϕ_3 are not greater than $n-1$ and n respectively. The constant C cannot vanish, since a relation

$$P = F_1 x(1-x)\frac{dP}{dx},$$

where F_1 is a single-valued function of x, cannot be verified by all the members of a P family. Division of (12) by C gives the desired representation for the Q function (10) by the expression (9). The same result follows if the Q function (11) is used instead of (10).

This representation holds even when one or more of the exponents in (6) is zero. To prove this we consider the families

$$x^\delta(1-x)^\epsilon P,$$
$$x^\delta(1-x)^\epsilon Q,$$

where P has the symbol (6) and Q the symbol (10) or (11), the numbers δ and ϵ being so chosen that in the new families no exponent is zero. For these functions (12) becomes, after division by C,

$$x^\delta(1-x)^\epsilon Q = \phi_2(x)x(1-x)\frac{d}{dx}[x^\delta(1-x)^\epsilon P] + \phi_3(x)x^\delta(1-x)^\epsilon P,$$

where ϕ_2 and ϕ_3 are polynomials in x of degrees not greater than $n-1$ and n respectively. If we carry out the indicated differentiation, rearrange terms and divide by $x^\delta(1-x)^\epsilon$, we have

$$Q = \phi_2(x)x(1-x)\frac{dP}{dx} + \{\phi_2(x)[\delta(1-x) - \epsilon x] + \phi_3(x)\}P,$$

which is of the form desired.

There remains the question whether we can dispense with the assumption that the families are irreducible. In my thesis I have shown that there are reducible Q families which do not have the same group as *any* P family. Obviously no such Q *family* (10) or (11) can be expressed in form (9), but particular members might be so expressed in terms of particular P functions. However, I have shown in my thesis that formula (3) holds for any three families having the same group, and (4) holds also when there are no semisingular points. Hence the representation we are discussing holds even in reducible cases provided Q and P have the same group and have no semisingular points.

3. *A Linear Expression for a Q Function in Terms of P Functions, with Constant Coefficients.* The other theorem, for which I have found no previous proof other than a count of constants, equally unconvincing, is that of Klein * which states that to form a Q function of order κ in the most general way we must make a linear combination, with constant coefficients, of $\kappa + 1$ kindred P functions. The statement is somewhat ambiguous; however, we shall now show that it is true, in the sense that *every* Q function is so expressible, with reservations in certain reducible cases as in § 2. This answers a question raised on page 49 of my thesis.

We will first suppose the Q family irreducible. If of even order we use the notation (10), if of odd order, (11), and express Q, by formulas (3) and (4), in terms of the two kindred P families, P and $\overline{P}$, where P has the symbol (6), and $\overline{P}$ the symbol

$$\overline{P}\begin{pmatrix}\alpha + 1, \beta, \gamma, \\ \alpha' - 1, \beta', \gamma', \end{pmatrix} x. \tag{13}$$

This gives us

$$CQ = \phi_2(x)\overline{P} + \phi_3(x)P, \tag{14}$$

where C is a constant, and ϕ_2 and ϕ_3 are polynomials in x of

* *Hypergeometrische Funktionen*, p. 233.

degrees not greater than $n-1$ and n respectively in case Q is of form (10), and of degrees not greater than $n-1$ each if Q is of form (11). The form of the exponents at zero shows that $\overline{P}$ cannot be equal to P multiplied by a single valued function F, hence C is not zero and we can divide through by C. A typical term of the product $\phi_3(x)P$ is

$$(15)\quad a_k x^k P\begin{pmatrix}\alpha, \beta, \gamma, x\\ \alpha', \beta', \gamma', \end{pmatrix} = a_k P_k\begin{pmatrix}\alpha+k, \beta-k, \gamma, x\\ \alpha'+k, \beta'-k, \gamma', \end{pmatrix},$$

where a_k is a constant; similarly the product $\phi_2(x)\overline{P}$ is a sum of terms $b_k\overline{P}_k$. Thus Q is expressed in the desired form

$$(16)\qquad Q = \sum_{k=0}^{n-1} b_k\overline{P}_k + \sum_{k=0}^{n} a_k P_k$$

if Q is of order $2n$, or

$$(17)\qquad Q = \sum_{k=0}^{n-1} b_k P_k + \sum_{k=0}^{n-1} a_k P_k$$

if Q is of order $2n-1$, where the a's and b's are constants and the functions $\overline{P}_k$ and P_k are P functions as indicated in (15).

As in § 2, we need additional hypotheses in case Q is reducible. Our proof is then not valid unless the families Q, P, $\overline{P}$ have the same group (which does not follow merely because corresponding exponents differ by integers) and no point is semi-singular. In my thesis (p. 59) I have considered the exceptional cases for a Q function of order 1, and have shown that such a Q function can always be expressed linearly in terms of two kindred P functions if its monodromic group is possible for P functions. This proof requires the examination of various types and classes of reducible groups, and we shall not attempt here to discuss the corresponding problem for Q functions of order higher than 1.

As a consequence of Klein's theorem it follows that every $\kappa+2$ irreducible Q functions of order κ having the same regular singular points and the same exponents at these points, but whose accessory parameters may differ, are linearly dependent; and this conclusion holds in certain reducible cases.

These theorems and methods of proof admit generalizations to the solutions of homogeneous linear differential equations of order n.

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THE REDUCTION OF SINGULARITIES OF PLANE CURVES BY BIRATIONAL TRANSFORMATION

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY, DECEMBER 28, 1922

BY G. A. BLISS

1. *Introduction.* In the domain of mathematics there are a number of famous theorems which have stimulated the keen interest of mathematicians over extended periods of time, and whose proofs have presented a long continued challenge to the powers of mathematical logic. To some of these, in spite of the fact that the theorems themselves remain unproved, our science is indebted for important advances.

As one illustration I may mention the well known theorem of Jordan which states that a simply closed continuous plane curve divides the plane into two and only two connected regions. This is a theorem of whose truth we are convinced at the start intuitively. But what we may for the moment call intuitive reasoning busies itself with simpler cases only, and is impatient of exceptions and refinements, unless by long study and continuous contemplation it has become a very sophisticated intuition indeed. Jordan was the first to insist that the theorem needed proof. Since his initial effort many others have attempted to give the conclusion of the theorem a substantial logical basis, and the theories of point sets and of functions of a real variable have been greatly enriched thereby. I am particularly interested to mention this theorem because it seems to me that with regard to it a satisfactory conclusion has been reached. There is doubtless still opportunity for improvements and simplifications in its proofs, but some at least of them have stood the test of examination by widely scattered experts. It is encouraging to have this evidence that not all of the mathematical questions generally recognized as most difficult are impossible to answer.

One should mention, of course, among the notable illustrations of the type of theorem which I have been discussing, Fermat's last theorem and the so-called four-color map

theorem. The latter asserts that every map can be colored with four different colors in such a way that each country has one color only, and no two countries adjoining along a boundary have the same color. No one has constructed a map which can not be so colored, and most of us, I think, believe that the theorem is true. But so far as I know no proof of it has as yet been widely accepted, the vote usually standing one, namely the author, strongly in favor, many others doubtful, and some, who have had the temerity to examine in detail the screed in question, definitely against. This state of affairs is even more pronounced in the case of Fermat's last theorem, with the difference that one may with justice, it seems to me, feel very uncertain of the validity of the theorem itself. All of the proved evidence is in its favor, but the behavior of very large numbers, far without the range of our every-day experience, is so beyond our ken that it should not be a surprise to us if some day it were found that the theorem is not true. We are fortunate in some ways, however, that these questions remain unanswered. For every attack by experts upon them is likely to yield, as has been true notably in the past, important advances in the domains of analysis situs and the theory of numbers. The chapter of progress associated with Jordan's theorem may still be subjected to improvements and additions, but as compared with those in which the centers of interest are the four-color map theorem and Fermat's last theorem, it is relatively completed and closed.

Many of you will have inferred from the title of this address the theorem of which I wish to speak to you especially to-day. It is the one which states that by a birational transformation an algebraic curve, no matter how complicated its singularities may be, can always be transformed into another having only double points with distinct tangents. It is with some diffidence that I designate this theorem as one of the type of which I have hitherto been speaking, since among my mathematical friends there are some for whose opinions I have great respect who disagree with me as to the difficulty of its proof. Unfortunately I have not been near enough to them to find out

with completeness what is in their minds. My purpose this afternoon will be achieved, however, if I can interest you again for the moment in some of the work of the sequence of distinguished mathematicians who have studied the theorem, and if I can make clear to those of you who have not specialized in the subject, some of the difficulties which they have sought to overcome. Up to the present time I do not know of any proof of the theorem which is sufficiently simple and unsophisticated to justify one at least of its important applications in the theory of algebraic functions to which I shall presently refer again.

2. *The Two Transformation Theorems.* In the statement of the theorem mentioned in § 1, reference is made to the notion of a birational transformation. There are two types of birational transformations, one of which is well illustrated by the very simple but important example $\xi = x$, $\eta = x/y$ relating the points of the xy-plane to those of the $\xi\eta$-plane. If one prefers homogeneous coordinates this transformation may be written in the form $\rho\xi_1 = x_2x_3$, $\rho\xi_2 = x_3x_1$, $\rho\xi_3 = x_1x_2$. It establishes a one-to-one correspondence between the points of the two planes with the exception of those on certain loci, in this case the coordinate axes, and is a special example of the general class of birational transformations which establish such correspondences between two planes and which are called Cremona transformations. Incidentally the transformation transforms every not too special algebraic curve $f(x, y) = 0$ of the xy-plane into a similar curve $\varphi(\xi, \eta) = 0$ of the $\xi\eta$-plane and establishes a one-to-one birational correspondence between their points. It is also possible, however, to have a one-to-one correspondence between the points of two curves $f(x, y) = 0$, $\varphi(\xi, \eta) = 0$, with coordinates (x, y) and (ξ, η) of corresponding points rationally expressible in terms of each other, but such that the correspondence is not a part of any Cremona transformation between the planes of the two curves such as has been described above. Such a transformation has been called a birational transformation relating the two curves alone. A simple example of such a transformation is defined by the

equations $\xi = x^2$, $\eta = y$ when applied to a curve not symmetric with respect to the y-axis.

Before proceeding to the discussion of the various proofs which have been made of the theorem I should like to say that there are really two closely related theorems which have been considered in this connection. To understand them we should remember that an "ordinary" singular point of a curve is one through which pass a finite number of branches whose tangents are distinct. For such a point $(x, y) = (a, b)$ in the finite part of the plane these branches will be determined analytically by convergent series of the form

$$x = a + \alpha t + \cdots, \qquad y = b + \beta t + \cdots,$$

in which the coefficients α and β are not both zero, and the different branches through (a, b) will have distinct ratios $\alpha : \beta$. The two theorems mentioned are then as follows.

THEOREM I. *Every irreducible algebraic plane curve can be transformed into another which has only ordinary singular points by a Cremona transformation of the planes of the two curves.*

THEOREM II. *Every irreducible algebraic plane curve can be transformed into another which has only ordinary double points by a transformation which is birational between the two curves.*

Theorem II is the one of which I have already spoken, and I may say here that there are two types of proofs of it. For one of them the conclusion of Theorem I is assumed, and a further birational transformation is then determined which will replace ordinary multiple points of multiplicity higher than two by ordinary double points. The other type of proof is more direct, presenting at once a birational transformation producing ordinary double points only, without the use of Theorem I.

In the theory of algebraic functions these two theorems have an important application. The genus of an algebraic curve, and the more important properties of the Abelian integrals and the field of rational functions associated with the curve, are invariant under birational transformations. The

presentation of the theory, from the standpoint of the so-called transcendental methods inaugurated by Riemann and developed and amplified by many other writers, is greatly simplified if one can presuppose that the curve under consideration has only ordinary singular points. Some further advantage in simplicity would be gained if only ordinary double points were present, but, as one of my mathematical colleagues has aptly said, this would be in the nature of a luxury rather than a necessity. It is significant, however, that most lecturers and writers who approach the subject from the standpoint of Riemann are content to start with the result of Theorem I, which they probably would not do if they found that Theorem II were provable in relatively as simple and elementary a fashion. Some of the other theories of algebraic functions which do not presuppose any special properties of the singular points of the basal curve seem to me highly interesting and desirable, notably that of Dedekind and Weber as amplified and presented by Hensel and Landsberg.* Theoretically this is certainly a very elegant and satisfying method of approach.

In speaking a few minutes ago of the correspondence established between two algebraic curves by a birational transformation, I used locution of which I very much disapprove, but which at that moment could not very well be explained. The elements of an algebraic curve which are set into one-to-one correspondence with those of another curve by a birational transformation are not the points of the curve, but the so-called branches or cycles. Through each finite point of the curve there passes one of these branches, if the point is simple, but possibly more than one if the point is multiple, and there are besides branches belonging to the so-called points at infinity. Analytically each branch is determined in non-homogeneous coordinates by equations of the form $x = P(t)$, $y = Q(t)$, where $P(t)$ and $Q(t)$ are convergent power series having usually only terms with positive powers of t, but for a finite number of branches having also a finite number of terms with negative exponents. If homogeneous coordi-

* *Theorie der algebraischen Funktionen einer Variabeln*, 1902.

nates are used the cycle or branch can always be represented by series having only positive powers of t. Weierstrass especially emphasized the point of view that a curve may well be thought of as the totality of the branches which satisfy its equation, and I think that failure to appreciate his contention has in some cases caused confusion in the proofs of the theorems of which I have been speaking. Birational transformations do not in general transform curves point for point into each other, but they do establish a unique correspondence between the branches of the two curves.

3. *Geometric Interpretations.* The most useful geometric picture of an algebraic equation $f(x, y) = 0$ which we have is its so-called Riemann surface. The effectiveness of the picture is due to the fact that there is a one-to-one continuous correspondence between the points on the surface and the branches which satisfy the equation. Furthermore the character of the cycle of sheets of which each point of the surface is a vertex indicates important properties of the corresponding analytic branch. Besides the Riemann surface, however, there are two other geometric representations of an algebraic curve which have especial importance in connection with the transformation theorems stated above, but each of which is open to the objection that it visualizes for us only the real branches of the algebraic curve in question.

For the first of these, consider the pencils of lines through two points A and B selected arbitrarily in the plane. In the pencil through A, for example, we may regard each ray as designated by the value x of its anharmonic ratio with three fixed rays through A chosen in advance; and the rays through B are similarly determined by numbers y. If we now associate with each pair of values (x, y) satisfying an algebraic equation $f(x, y) = 0$ the point of intersection of the corresponding x- and y-rays through A and B, we obtain a curve representing geometrically the real branches of our equation. In order to give equal consideration to all the rays through A and B without using infinite values for the coordinates we may introduce the homogeneous variables x_1, x_2, and y_1, y_2 defined by the equations $x = x_1/x_2$, $y = y_1/y_2$.

The other geometric representation of the equation $f(x, y) = 0$ is so familiar that I need not speak of it in detail. If we replace x and y by homogeneous coordinates x_1, x_2, x_3 defined in the usual way by the equations $x = x_1/x_3, y = x_2/x_3$, then the equation $f(x, y) = 0$ determines a curve in the projective plane as soon as a coordinate triangle and unit point have been assigned.

For the sake of convenience let us agree to call the former of these two geometric interpretations of the equation $f(x, y) = 0$ the function-theoretic one, and the latter simply the geometric interpretation. The reason for this nomenclature is that the former is closely allied to the theory of algebraic functions, while the latter is that of ordinary plane projective geometry. The function-theoretic interpretation is the more convenient one for representing properties of curves which are invariant under transformations of the form

$$(1) \qquad x = \frac{a_1\xi + a_2}{a_3\xi + a_4}, \qquad y = \frac{b_1\eta + b_2}{b_3\eta + b_4},$$

while the geometric interpretation has similar advantages for transformations

$$(2) \qquad x = \frac{a_1\xi + b_1\eta + c_1}{a_3\xi + b_3\eta + c_3}, \qquad y = \frac{a_2\xi + b_2\eta + c_2}{a_3\xi + b_3\eta + c_3}.$$

We say in either case that a point of the curve $f(x, y) = 0$ with finite or infinite coordinates is an ordinary double point if it has this character after an appropriate transformation (1) or (2) into a non-special point with finite coordinates. The points on the line AB in the function-theoretic representation are special points.

It is perhaps more satisfactory to define analytically what is meant by a curve with ordinary double points only. In the function-theoretic case the equations of a branch can always be taken in the form

$$x_i = a_i + \alpha_i t + \cdots, \qquad y_i = b_i + \beta_i t + \cdots, \qquad (i = 1, 2),$$
$$(a_1, a_2) \neq (0, 0), \qquad (b_1, b_2) \neq (0, 0),$$

and the branch is said to be linear if the two determinants

$$a_1\alpha_2 - a_2\alpha_1, \qquad b_1\beta_2 - b_2\beta_1$$

do not both vanish. A second branch with primed coefficients has the same center if

$$a_i' = ka_i, \qquad b_i' = lb_i, \qquad (i = 1, 2);$$

and two branches with the same center are said to have distinct tangents if the expression

$$(a_1\alpha_2' - a_2\alpha_1')(b_1'\beta_2 - b_2'\beta_1) - (a_1'\alpha_2 - a_2'\alpha_1)(b_1\beta_2' - b_2\beta_1')$$

is not zero.

In the geometric case a branch has equations

$$x_i = a_i + \alpha_i t + \cdots, \quad (i = 1, 2, 3), \ (a_1, a_2, a_3) \neq (0, 0, 0),$$

and is linear if the determinants $a_i\alpha_k - a_k\alpha_i$ $(i \neq k)$ are not all zero. A second branch with primed coefficients has the same center if $a_k' = \rho a_k$ $(k = 1, 2, 3)$; and two branches with the same center have distinct tangents if the determinant $(a_1\alpha_2\alpha_3')$ is different from zero. For branches with finite centers $(x, y) = (a, b)$ these definitions of branches with distinct tangents reduce to those of the usual cartesian analytic geometry. A curve with no singularities except ordinary double points is in either interpretation one such that all of its branches are linear, such that none of its points is a center of more than two of its branches, and such that every pair of its branches having a common center has distinct tangents. These properties are invariant in the function-theoretic and geometric cases under the respective transformations (1) and (2).

Now it happens that the branches which satisfy an equation $f(x, y) = 0$ may determine a curve with ordinary double points only in one of these interpretations, while more complicated singularities are present in the other. This is due to the fact that we have added in two different ways points with infinite coordinates to the totality of finite points (x, y) in the plane in order to obtain the two interpretations. If, for example, the line $x = \infty$ through A is intersected by n distinct branches in the function-theoretic case, then the curve for the geometric interpretation of the equation will have an ordinary singular point with n branches at the intersection of the x-axis with the line at infinity. Similarly if the line at

infinity in the projective plane intersects the geometric curve in n points distinct from each other and from the x- and y-axes, then the function-theoretic curve will have an ordinary singular point with n branches through the intersection of the lines $x = \infty$ and $y = \infty$.

We could readily analyze this correspondence in greater detail, but the essential thing to remark is that there is really a pair of Theorems II which are not equivalent. The distinction between these theorems has not, in my opinion, been sufficiently emphasized in the literature, though proofs applying to both have been presented. At the present time the function-theoretic theorem appears to me somewhat easier to handle, though neither of them is simple.

4. *Noether's Theorem.* Those of you who have not been specialists in this domain will perhaps be interested to know something of the origins of these theorems. Theorem I is, with what seems to be universal agreement, ascribed to Noether. In 1871 * he stated the theorem and indicated the proof which has since been modified and elaborated in one way or another by many writers. A difficulty in the proof not explained in this original paper can be removed by methods used in the same year by Hamburger † in his study of the Puiseaux expansions at a singular point. Noether ‡ himself in 1876 treated the question more elaborately in a paper devoted to the analysis of the singular points of an algebraic curve. Bertini § was dissatisfied with the methods used by Noether, and in 1888 gave a direct proof of the theorem which he characterizes as simple and rigorous. The proof with which I am most familiar is that given by Picard || and ascribed by him to Simart. The geometers would be perhaps most interested in those of Severi ¶

* Göttinger Nachrichten, 1871, page 267.

† Zeitschrift für Mathematik und Physik, vol. 16 (1871), p. 461.

‡ Mathematische Annalen, vol. 9 (1876), p. 166, especially § 5.

§ Lombardo Rendiconti, (2), vol. 21 (1888), page 326. In commenting on Noether's paper he says "le considerazioni (§ 5) lasciano forse a desiderare maggiore semplicita e chiarezza," and he says that his intention in his own paper is to prove Noether's theorem "semplice e rigorosa."

|| *Traité d'Analyse*, vol. 2, 1st ed., 1893, page 360; 2d ed., 1905, p. 404.

¶ *Lezioni di Geometria Algebrica*, 1908, p. 61.

and Enriques,* but there are many others.† The proofs are all modifications of Noether's original one, and there is no question with regard to the validity of the theorem. I may add that Halphen ‡ gave two methods of transforming a curve into one with ordinary singular points only by birational transformations which are not Cremona transformations.

The method of the proof is to place the origin of coordinates at one of the singular points of the curve, say one of order k, and then to apply the simple quadratic transformation $\xi = x$, $\eta = x/y$ of which I have spoken above. In the new curve the character of the singular points not at the origin remains the same as before, but the singularity of order k at the origin is exploded by the transformation, and when one examines the pieces remaining two new ordinary singular points are found at corners of the coordinate triangle, plus one or more other fragmentary singular points on one side of this triangle the sum of whose orders is k. The difficulty which I have mentioned above and which was not originally explained by Noether lies in the possibility that there may be only one such fragment of the same order as the original singularity, in which case the situation might easily be as bad after as before the transformation. It is provable, however, that after a finite number of repetitions of the transformation the number of the fragments will surely be greater than one, and their orders each less than k. By a continuation of the process, therefore, one can finally replace the singular point originally placed at the origin, and indeed all of the singular points, by ordinary singular points only. It is not surprising that there have been so few variations from the method of proof originally

* *Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche*, vol. 2, 1918, p. 417.

† See ENCYCLOPÄDIE, III C 4, p. 363; Pascal, *Repertorium der Höheren Mathematik*, vol. 2, 2d ed., p. 291; Brill und Noether, *Die Entwicklung der Theorie der algebraischen Functionen*, JAHRESBERICHT DER VEREINIGUNG, vol. 3 (1892–3), p. 369. Interesting references which do not seem to be listed in these places are Jordan, *Cours d'Analyse*, vol. 1, 2d ed., 1893, p. 588; and Clebsch-Lindemann, *Vorlesungen über Geometrie*, vol. 1, 1876, p. 491.

‡ COMPTES RENDUS, vol. 80 (1875), p. 638, and OEUVRES, vol. 1, p. 358; JOURNAL DE MATHÉMATIQUES, (3), vol. 2 (1876), p. 87.

proposed by Noether, if one remembers the theorem that every Cremona transformation is decomposable into a succession of projective transformations and simple quadratic transformations of the type we have been considering.

5. *The Kronecker Group of Proofs of Theorem* II. With regard to Theorem II, it seems to me that the story is quite a different one. The origin of the theorem is less easy to determine, and no proof of it has been adopted with universal or even with widespread approval. During a somewhat cursory examination of the literature over a year ago I found so many doubtful comments by authors themselves upon proofs which had preceded theirs, and so many questions of my own which it seemed difficult to answer, that I resolved to try by a closer personal examination to convince myself of the merits of some of the published demonstrations of the theorem. It is of the impressions gained from this study that I wish to speak to you now. After we have looked for a moment at the various types of proofs which have been given it will perhaps be easier to describe intelligibly the sources from which the theorem seems to have been developed.

The proofs fall into four main groups,* and I shall speak first of that which originated with Kronecker because it is the one in which I have myself been primarily interested. In 1881 Kronecker † published a paper in which he discusses the factorization of the discriminant of an algebraic function, and which he had presented to the Berlin Academy many years before, in 1862. It is a very well known paper, for many reasons one of the most suggestive and interesting which I have ever read. In it he shows that the discriminant of an algebraic function y of x, defined by an algebraic equation $f(x, y) = 0$, has always the form $D = R^2\Delta$, where R and Δ are two polynomials in the variable x. If the equation $f(x, y) = 0$ is subjected to a birational transformation of the form

$$\xi = x, \qquad \eta = r(x, y)$$

* For the literature see Brill and Noether, loc. cit., pp. 369 ff.; Wirtinger, ENCYCLOPÄDIE, II B 2, p. 127; Berzolari, ibid., III C 4, pp. 362 ff.; and in Pascal's *Repertorium*, loc. cit.

† JOURNAL FÜR MATHEMATIK, vol. 91 (1881), p. 301.

upon the dependent variable alone, then the resulting equation $\varphi(x, \eta) = 0$ will have a discriminant $D_1 = R_1^2\Delta$ in which R_1 is in general different from R, while the factor Δ remains unchanged. A principal purpose of the paper is to determine the birational transformation in such a way that the roots of R_1 are distinct from each other and from those of Δ. If this has been accomplished, then it is provable that the curve $\varphi(x, \eta) = 0$ can have no singularities in the finite part of the $x\eta$-plane except ordinary double points. This last step I do not find explicitly explained in Kronecker's paper, though it is clear from other sources that he and his followers were well aware of the geometric significance of his transformation of the discriminant.

Kronecker states that the results of the paper of which I have been speaking were in part known to him as early as 1857, and that he used them repeatedly in his lectures for many years before the paper itself was published. His transformation of the discriminant was adopted by Weierstrass in lectures on the theory of algebraic functions in 1869. I am acquainted with Weierstrass' treatment of the question only through a brief outline by Brill and Noether* and a more detailed paper based upon this outline published by Thomé in 1903.† The papers of Kronecker and Thomé are both very clear and convincing, but neither of them affords a complete proof of either the function-theoretic or the geometric Theorem II because after their transformations there may still be complicated singularities at infinity, though only ordinary double points in the finite part of the plane.

In order to secure his birational transformation simplifying the discriminant, Kronecker made use of the rational functions $\eta(x, y)$ on the Riemann surface of the curve $f(x, y) = 0$ which have no poles on the finite part of the surface, the so-called integral algebraic functions in the field of rational functions associated with the curve. A general formula for such a function is

$$(3) \qquad \eta(x, y) = u_1(x)\eta_1(x, y) + \cdots + u_n(x)\eta_n(x, y),$$

* Loc. cit., pp. 375–6.

† JOURNAL FÜR MATHEMATIK, vol. 126 (1903), p. 52.

where $\eta_1, \cdots, \eta_n$ is a suitably selected set of linearly independent functions of this sort, and the coefficients $u_1, \cdots, u_n$ are arbitrary polynomials in x. If these coefficients u_k are suitably specialized, then x and η will satisfy an algebraic equation $\varphi(x, \eta) = 0$ birationally related to $f(x, y) = 0$ and with a discriminant of the desired simplicity.

In 1902 Hensel and Landsberg * extended this reasoning of Kronecker. The functions $\eta(x, y)$ which Kronecker uses are restricted to have minimum orders zero at the finite points of the Riemann surface of $f(x, y) = 0$, but are unrestricted at the infinite points. A formula like (3) still holds, however, if the functions $\eta(x, y)$ have the properties just mentioned with the exception that minimum orders different from zero, either positive or negative, are admitted at a limited number of finite points of the Riemann surface. If now the number of poles allowed for $\eta(x, y)$ is sufficiently large, and if the coefficients $u_1, \cdots, u_n$ are again suitably specialized, the function $\eta(x, y)$ will not only satisfy an equation $\varphi(x, \eta) = 0$ birationally related to $f(x, y) = 0$, and have a discriminant of the form attained by Kronecker, but it will also have finite distinct values at the infinite points of the Riemann surface. This brings us to the function-theoretic Theorem II provided that the poles allowed for $\eta(x, y)$ are of order one and over distinct values of x. For if on the Riemann surface of $\varphi(x, \eta) = 0$, which is the same as that for $f(x, y) = 0$, the function $\eta(x, y)$ has distinct finite values at the value $x = \infty$ and simple poles over distinct finite values of x, then the infinite branches of $\varphi(x, \eta) = 0$ give rise to no multiple points in the function-theoretic interpretation of the equation. If furthermore the discriminant of $\eta(x, y)$ has the simple form attained by Kronecker, then the curve $\varphi(x, \eta) = 0$ can have no singularities at finite points (x, η) except ordinary double points.

The discussion of the separation of the roots of the discriminant given by Hensel and Landsberg seems to me at one point incomplete † without the addition of some

* *Theorie der algebraischen Funktionen einer Variabeln*, pp. 402–9.

† Loc. cit., the paragraph on pages 407–8.

details from Kronecker's original memoir. Their final statement of the theorem does not distinguish between the function-theoretic and geometric cases upon which I have insisted above, and it is followed by the assertion without a reference that the theorem was first proved by Kronecker. So far as I can discover from published papers, this last statement is not quite just to themselves. The extension of Kronecker's proof which enables one to secure proper behavior of the transformation at infinite as well as finite points of the Riemann surface is apparently their own, and it is a much more difficult extension than is indicated by my brief outline of their argument.

I have been interested to attempt to prove the geometric Theorem II also by Kronecker's method, and have found a proof which seems to me satisfactory. If an algebraic equation $\varphi(\xi, \eta) = 0$ of degree q defines q expansions for η at $\xi = \infty$ of the form

$$\eta = \alpha_k \xi + \beta_k + \gamma_k \frac{1}{\xi} + \cdots, \qquad (k = 1, \cdots, q; \alpha_k \neq \alpha_l \text{ for } k \neq l),$$

and if the discriminant for η has the simplified form attained by Kronecker, then it is provable that in the projective plane the corresponding curve meets the line at infinity in distinct points and has no singularities in the finite part of the plane except ordinary double points. To obtain such a curve birationally related to $f(x, y) = 0$ one can start by selecting $q > 2p + 2$ ordinary points on the Riemann surface of $f(x, y) = 0$ over distinct values of x, p being the genus of the surface. The functions $\eta(x, y)$ which have no singularities except possibly simple poles at these points constitute a linear family of the form

$$(4) \quad \eta(x, y) = c_1\eta_1(x, y) + \cdots + c_\nu\eta_\nu(x, y), \qquad (\nu = q - p + 1),$$

the coefficients c_k being constants. A suitable special choice of these coefficients can easily be made so that the resulting function $\xi(x, y)$ has surely a simple pole at each of the q points. With more difficulty a second special function $\eta(x, y)$ of the family can then be chosen which with ξ satisfies an equation $\varphi(\xi, \eta) = 0$ birationally related to $f(x, y) = 0$ and which has the further properties described above.

Here, then, based upon the methods of Kronecker, are proofs of both the function-theoretic and geometric Theorems II. They are what I should call sophisticated proofs, since they make use of properties of rational functions with prescribed poles on the Riemann surface of $f(x, y) = 0$, which are well known and well established but not elementary. These properties are proved in the theory of algebraic functions as presented by Hensel and Landsberg without restrictive assumptions upon the singularities of $f(x, y) = 0$. If the so-called transcendental method of Riemann is adopted, however, it is usually presupposed in proving them that the singularities of the curve $f(x, y) = 0$ are ordinary multiple points only, or, in other words, that the result of Theorem I has been attained by transformation. An easily derived property which I have found for the linear families (4) aids very much in simplifying the proofs of both the function-theoretic and geometric interpretations of Theorem II.

6. *The Halphen Group of Proofs of Theorem* II. There is a second very interesting series of discussions of Theorem II which seems to have been inaugurated by Halphen in 1884.* So far as I have been able to discover this paper of Halphen's contains the first explicit statement and proof of Theorem II which has been published. In 1893 Picard † reproduced Halphen's argument. At its conclusion, in commenting on a particular step in the proof, he says that the point is not to be doubted, but that he does not see quite the means of establishing it so as to avoid every objection. The reasoning by continuity which he has just been using makes the accuracy of the conclusion seem to him more than probable. In a footnote he adds that Halphen arrives at the theorem without mentioning the difficulty.

This comment of Picard seems to have instigated a series of attacks on the theorem which continued for more than a

* *Étude sur les points singuliers des courbes planes algébriques*, appendix to Salmon's *Traité de Géométrie Analytique* (*Courbes Planes*), translated by O. Chemin, edition of 1903, pp. 627–31.

† *Traité d'Analyse*, vol. 2, 1893, p. 364.

decade. In the same year, 1893, Simart * modified Halphen's proof as presented by Picard and remarks that the theorem, though affirmed by many authors, has never to his knowledge been demonstrated rigorously, and he offers, therefore, a demonstration which he believes to be secure against every objection. Only a little later in the same year Poincaré † proposed a geometric proof of a different sort, of which he says that the two principal steps will possibly seem almost evident. Yet there will perhaps be some interest, he continues, in having a demonstration which is secure against every objection. In 1895 Appell and Goursat ‡ refer in a footnote to the proofs of Simart and Poincaré, but rely for their establishment of the theorem upon another very interesting transformation previously used by Halphen § to secure a curve with ordinary singular points only. In 1896 Vessiot ‖ states that many demonstrations of the theorem have been given, but that the modification of Appell and Goursat's proof which he is about to explain will not be without interest on account of its simplicity and complete rigor. In a footnote ¶ he says that if one accepts as evident the steps (*e*) and (*f*) of his demonstration, as Appell and Goursat have done at the analogous points of their proof, one may arrive at his conclusion more rapidly. But the explanations which he gives seem to him indispensable if one desires to establish these two points beyond all doubt.

It seems to me that these comments express considerable dissatisfaction and uncertainty with regard to the proofs of Theorem II which were at that time under discussion. The transformations suggested by Halphen are most interesting and plausible, and one would expect, as Picard implies, that their effectiveness can be conclusively established. I do not feel convinced as yet that this has been done. Simart's paper is so concise that I have so far been unable to verify all the

* Comptes Rendus, vol. 116 (1893), p. 1047.

† Comptes Rendus, vol. 117 (1893), p. 18.

‡ *Théorie des Fonctions Algébriques*, 1895, p. 282.

§ Journal de Mathématiques, loc. cit.

‖ Annales de Toulouse, vol. 10 (1896), p. D1.

¶ Page D6.

steps of his argument to my own satisfaction. Vessiot uses a combination of projective transformations with the simple transformation

$$\xi = x, \qquad \eta = \frac{dy}{dx} = -\frac{f_x}{f_y}.$$

One cannot in general by a preliminary projective transformation prevent this function η from having poles in the finite part of the Riemann surface of $f(x, y) = 0$. Each of these poles provides a branch which in the projective $x\eta$-plane passes through the intersection of the η-axis and the line at infinity. A singular point is therefore introduced at infinity on the projective curve which cannot be made to disappear by a subsequent projective transformation. It does not seem possible, therefore, to prove the geometric Theorem II by this method. On the other hand I am inclined to believe that Vessiot's method may afford a relatively simple proof of the function-theoretic Theorem II. By a projective transformation one should be able to bring it about that $\eta = dy/dx$ has distinct finite values at the infinite points of the Riemann surface, and at most simple poles over distinct values of x in the finite parts of the surface. These properties insure the existence of simple points only at the intersections of the function-theoretic curve with the rays $x = \infty$ and $y = \infty$ through the points A and B. The methods of Vessiot could then be used to secure at most ordinary double points at finite points (x, y). Some of the steps are, however, analytically delicate.

7. *The Bertini Group of Proofs of Theorem* II. A third group of proofs, this time of the geometric Theorem II, is due to Bertini.* In 1891 he was apparently unaware of Halphen's paper of 1884, for he states that in many works on geometry Theorem II is applied, but that as far as he knows no one has given an explicit demonstration. He proposes, therefore, one which appears to him to be simple and rigorous. It seems likely that the form which he devised was suggested by some studies of birational transformations of curves published by Clebsch and Lindemann in 1876.† In 1894 Bertini republished

* Rivista di Matematica, vol. 1 (1891), p. 22.

† *Vorlesungen über Geometrie*, 1876, pp. 661 ff.

his proof in the MATHEMATISCHE ANNALEN,* and in a footnote explained that a geometric difficulty which Poincaré had avoided somewhat circuitously in 1893 had already been discussed by himself. Klein added a second footnote in which he suggested an interesting geometric transformation, equivalent to Bertini's, which he says had been communicated to him orally by Clebsch in 1869. In 1905, in the second edition of the second volume of his *Traité d'Analyse*,† Picard adopts Bertini's proof with slight modifications only. In 1906 Walker ‡ remodeled Bertini's method in detail in accordance with Klein's geometric suggestion. In Walker's introduction he refers to the proofs listed in this and the preceding groups, and remarks that they are written in very concise style and leave a great many minor points to the reader. In 1907 W. H. and G. Ch. Young § described a new condition on Bertini's transformation which must be added in order to make it entirely satisfactory. They were evidently unacquainted with Walker's thesis, in which this objection had been answered in a different way but had not been pointed out explicitly.

The great advantage of the form of Bertini's proof adopted by Walker lies in the fact that the various steps of the proof have geometric interpretations in ordinary spaces of two and three dimensions. It is consequently easier by this method to classify the properties which must be imposed upon Bertini's transformation, and to make sure that all of them have been suitably secured. The proof is in outline as follows. Through six points of a plane π, not on a conic and no three collinear, there pass four linearly independent cubic curves $f_i(x_1, x_2, x_3)$, $(i = 1, \cdots, 4)$. The equations

$$\rho y_i = f_i(x_1, x_2, x_3), \qquad (i = 1, \cdots, 4) \tag{5}$$

define a cubic surface F in the three-dimensional y-space, and the points of F are in one-to-one correspondence with those of the plane π with the exception that six of the straight lines

* Vol. 44 (1894), p. 158.

† Page 408.

‡ *On the resolution of higher singularities of algebraic curves into ordinary nodes*, Dissertation, Chicago, 1906.

§ TORINO ATTI, vol. 42 (1907), pp. 82–6.

on the surface correspond to the six base points. For each of these lines there is, however, a one-to-one correspondence between its points and the rays through the corresponding base point. Let K be an irreducible algebraic curve in the x-plane with ordinary singular points only, and let the base points of the transformation just described be selected so that one of them is at a singularity A of K but the others not on K. Then the curve K is transformed by equations (5) into a new curve K' on the surface F, and K' has singularities similar to those of K with the exception that one of them, namely A, is dispersed into ordinary points. If a suitable center of projection O is now taken on the surface F, the curve K' can be projected into a plane curve K'' which has possibly new ordinary double points in place of A, but whose other singularities are the same in number and in type as those of K. By a succession of such transformations all of the singular points of K may therefore be replaced by ordinary double points.

The difficult part of the demonstration is the proof that it is possible to choose properly the center of projection O. If O is selected on F but distinct from K' and the straight lines of F, and distinct from the cones generated by K' at its own singular points, then the projection of K' from O defines a curve K'' in one-to-one correspondence with K' except for a finite number of double points of K'' where the correspondence is one-to-two. The choice of O in this manner on the cubic surface F effectively prevents any ray through O from being a trisecant of K'. If O is not on the developable of the tangents to K', then K'' will have no cusps. If O is not on the tangent planes to F at the multiple points of K', and not on the cones generated by K' at its own singular points, then each multiple point of K' will project into an ordinary multiple point of the same character for K''. Finally if O is distinct from the ruled surface whose generators are lines joining points of K' with coplanar tangents, then the only double points introduced for K'' will be ordinary double points. In order to make the proof complete it must be shown that each of the loci just mentioned exists and can be avoided in the

selection of the point O. Walker has not only done this but he has also discussed in detail the properties of the cubic transformation from the plane π to the surface F. His thesis fills all told about fifty pages, a decided contrast to Bertini's original outline of slightly more than one page. I have been satisfied, by Walker's reasoning or by arguments of my own which seemed to me more convenient, that all of his steps are well justified. It would be highly desirable to have the proof shortened and simplified without sacrificing clearness or accuracy, and I am inclined to think that it could be done.

8. *Other Geometric Proofs of Theorem* II. There is a fourth and final principal group of proofs of Theorem II, more geometric in character, with which I am relatively much less familiar.* The plan underlying most of them is to transform the given plane algebraic curve into a twisted curve in a higher space devoid of singularities, and then to project this non-singular curve upon a two-dimensional plane in such a way that only ordinary double points remain. Intuitively the possibility of these two steps, as Poincaré says, does seem almost evident, and it is perhaps for this reason that in most of the geometric proofs the final projection is hardly mentioned. It seems to me, however, that this projection is a really difficult part of the proof. Poincaré's proof, for example, is very condensed, but all of his steps can be verified without serious difficulty up to his discussion of the projection of the non-singular curve in higher space into a non-singular curve in three-space. At this point his reasoning seems to me much too concise, and even questionable. In Walker's form of Bertini's proof the projection of the space curve K' on the surface F into a plane curve K'' is the most difficult step. My doubts about the ease of making the projection have been strengthened by

* Veronese, MATHEMATISCHE ANNALEN, vol. 19 (1881), p. 213; Poincaré, loc. cit.; del Pezzo, NAPOLI RENDICONTI, (2), vol. 7 (1893), p. 15; Vessiot, BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE, vol. 22 (1894), p. 208; Pieri, RIVISTA DI MATEMATICA, vol. 4 (1894), p. 40; Segre, ANNALI DI MATEMATICA, (2), vol. 25 (1897), p. 43; Severi, *Lezioni di Geometria Algebrica*, 1908, p. 172; Del Re, MODENA MEMORIE, (2), 10 (1894), p. 447, footnote; Jamet, Nouvelles Annales, (3), vol. 19 (1900), p. 506.

some further interesting evidence. After I had made the proof of the geometric Theorem II, following Kronecker, I read Severi's geometric proof and was surprised to find a close analogy between my own argument and his. This was a result of the relationship between the theory of rational functions of x, y with prescribed poles on the Riemann surface of $f(x, y) = 0$ and the geometric theory of the groups of points intersected on the curve $f(x, y) = 0$ by linear families of curves. It is these linear families which Severi uses to get his transformation from the plane curve to a non-singular curve in higher space. But Severi relegates the question of the projection of his space curve upon a plane, so as to leave only ordinary double points, to a short footnote containing only some remarks on how to avoid trisecants, whereas in the Kronecker theory the equivalent steps form the most delicate part of the proof. For the present, then, I am discontented, to say the least, when I find a geometric proof which does not explain in detail the projection mentioned. The transformation from the original plane curve to the non-singular space curve is relatively easy and can be done in a number of ways.

9. *Origin of Theorem* II. You will understand, I think, from what has been said above, that it is not easy to designate with certainty the origin of Theorem II. Halphen in 1884 ascribed the theorem to Noether, and he has since been imitated by a number of other writers, probably on account of the close relationship between Theorem II and Noether's Theorem I. On the basis of published papers it would be difficult to establish this origin. Klein, after stating the theorem without proof in the text of his *Riemannsche Flächen*,* inquires in a footnote "Wo ist der Satz zum ersten Male gedruckt?", and states that he himself first heard it orally from Kronecker in the autumn of 1869. Hensel and Landsberg also ascribe the theorem to Kronecker. If we rely on printed memoirs, however, one must admit that Kronecker's paper, first published in 1881, was devoted to the analysis of the discriminant of an algebraic function and was essentially analytic in character. His

* 1892, p. 245.

reduction has indeed an important bearing upon Theorem II, but it affords only a partial proof. You will remember that Klein, in the footnote which he added to Bertini's ANNALEN paper of 1894, states that the geometric transformation there described was communicated to him orally by Clebsch in 1869. Brill and Noether* further publish an extract from a letter of Klein to Noether dated December 17, 1869, in which Klein describes two results attained, respectively, by Kronecker and Clebsch. The first is the transformation of an arbitrary algebraic plane curve into another with no singularities except a single ordinary singular point; and the second is the transformation of a curve with a single ordinary singular point into one with at most ordinary double points by the geometric process which Klein ascribes to Clebsch in his footnote to Bertini's paper. Klein further adds in his letter that he believes Clebsch's method can be extended so as to reduce any number of arbitrary singularities to ordinary double points. Brill and Noether contend, and it seems to me with justice, that these assertions would be more effective as the basis of a claim for priority in the proof of the theorem if they had been afterwards substantiated in print. In 1897 Segre,† speaking of the transformation of a plane curve into one in higher space with no singularities, says that solutions of the problem are contained implicitly in a paper by Brill and Noether of 1874,‡ and more explicitly in Veronese's paper of 1881 previously cited. As I said above, however, the first explicit published statement of the theorem with proof was probably that of Halphen in 1884. On the whole, I am inclined to agree with the conclusion implied by an above quoted statement of Bertini in 1891. The theorem seems to have gradually established itself in the consciousness of mathematicians during the decades from 1860 on, and an assignment of it to a definite origin would probably be incorrect. It is without doubt true that many a theorem now confidently attributed to a particular

* JAHRESBERICHT, loc. cit., p. 371.

† Loc. cit., p. 43.

‡ MATHEMATISCHE ANNALEN, vol. 7 (1874), p. 269.

mathematician would upon closer examination prove rather to be, as this one seems to be, the product of a gradual evolution in the minds of many individuals.

It seems to me that there is still a need for concise and clear proofs of the function-theoretic and the geometric Theorems II. The Hensel and Landsberg proof of the former, and Walker's proof of the latter following Bertini and Klein, are the only published ones with whose details I am at the present time personally well content. There are objections to both of them. It would be very difficult and inconvenient to analyze and understand the former unless one had read many of the preceding pages of Hensel and Landsberg's book, and possibly also Kronecker's original memoir. Walker's proof has the sort of geometric background which should enable one to decide definitely whether or not all essential details have been thought of and suitably discussed. But it is very long, and, having been published privately, it has not been subjected to the close scrutiny of mathematicians in various centers which seems to me desirable. The last objection applies also to my own proof of which I spoke above. So far as I know there is at present only one person who has definitely, though somewhat timidly, expressed his approval of this effort of mine, and his identity I must leave you to surmise. In conclusion I have only one further hope to express, namely, that no new proof will appear in the very condensed style which makes it almost impossible to decide whether the theorem has really been established or not. There are a number of these already, and the accumulation of others would only add to the difficulties of every earnest searcher for the truth in this domain. It seems to me that brevity at the expense of clearness, in a situation of this sort, should not be mistaken for simplicity.

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SHORTER NOTICES

James Stirling: A Sketch of his Life and Works, along with his Scientific Correspondence. By Charles Tweedie. Oxford, Clarendon Press, 1922. x + 213 pp.

This book is an interesting and valuable contribution to the history of mathematics, giving as it does a considerable amount of new information about the work of the early successors of Newton. One of the ablest of these was James Stirling, and thus his hitherto unpublished mathematical correspondence contains much of interest.

The book is divided into three parts, "Life," 22 pages, "Works," 26 pages, and "Correspondence," 160 pages. The first part adds some interesting details to the rather meagre stock of information available concerning Stirling's life. He was one of the non-juror students at Oxford, and as such under suspicion at the time of the accession of George I, when there were riots and a revolution in favor of the restoration of the Stuarts seemed imminent. In fact, because of his Jacobite leanings, Stirling was deprived of a scholarship which he had previously held; but the usual account * that he was expelled from the University and "fled to Italy" is incorrect, as the author brings out the fact that Stirling only left Oxford in 1717, to go to Venice on the offer of a mathematical professorship there. This was the year in which he published (in Oxford) his book on cubic curves. The offer in Venice was eventually declined, on account of the religious conditions attached to it. He remained in Italy, however, till about 1724, when he located in London. He was a friend of Newton and of Maclaurin, and was highly esteemed by other contemporary mathematicians, particularly Machin, DeMoivre, and Cramer. The latter part of his life (from the age of 43 on) he devoted to commercial work, and thus his mathematical productivity came to an end just at the time when he was in a position to have added much to the development of the subject.

Stirling's chief works are *Lineae Tertii Ordinis* (1717) and *Methodus Differentialis* (1730). These are well described in the second part of the book under review. The *Methodus Differentialis* contains most of Stirling's own contributions to mathematics; these relate to the summation of series, to interpolation, and to approximation formulas, the most famous of the last being "Stirling's Formula,"

$$n! = \sqrt{2n\pi} \cdot n^n \cdot e^{-n}.$$

He finds the exterpolated value of the term preceding the first in the series of factorials,

1, 1, 2, 6, 24, 120, etc.

to be 1.7724538502, and says that this is equal to $\sqrt{\pi}$. He also introduces the Beta Function as an integral, for the interpolation of

$$a, \quad \frac{r}{p}a, \quad \frac{r(r+1)}{p(p+1)}a, \quad \cdots,$$

* *Dictionary of National Biography*; Cantor, *Geschichte der Mathematik*, vol. 3, p. 372.

and obtains a result equivalent to

$$\frac{B(p+n',q)}{B(p'q)} = \frac{p(p+1)\cdots(p+n-1)}{(p+q)(p+q+1)\cdots(p+q+n-1)}.$$

The third part of the book is the most important, both in volume and in the additional light which it throws upon Stirling's work in its relation to his contemporaries. Thus there are 11 letters from Maclaurin to Stirling, 10 from Cramer, 3 from Nicholas Bernoulli (1687–1750), the nephew of John Bernoulli, 2 from Machin, and one, perhaps the most valuable in the entire collection, from Euler. There are 9 letters written by Stirling (4 to Maclaurin, 1 to Cramer, 1 to N. Bernoulli, 1 to Castel, 1 to Bradley, and 1 to Euler). All the letters to Stirling testify to the high regard in which the writer held him, and to the influence which his writings exerted upon contemporary mathematics. The letter from Euler mentioned above, contains a rich fund of information about his work in series, including $\Sigma_{n=0}^{\infty} 1/n^k$, $\Sigma_{n=0}^{\infty} 1/(2n+1)^k$, and others. A curious infinite product which he gives is perhaps not generally known:

$$\frac{3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdots}{4\cdot 4\cdot 8\cdot 12\cdot 12\cdot 16\cdot 20\cdot 24\cdot 28\cdots},$$

where the factors of the numerator are the successive prime numbers, and each factor of the denominator is the integer of form $4k$ nearest to the corresponding factor in the numerator. Euler says the limit of this fraction is $\pi/4$, but does not give any hint as to his method of proof.

Mr. Tweedie has done a valuable piece of work, which will be found useful not only as a contribution to the history of mathematics, but also as giving in very readable form a source of information as to one of the founders of the modern theory of finite differences.

R. B. McClenon

The Fourth Dimension and the Bible. By W. A. Granville. Boston, Richard G. Badger, 1922. 9 + 119 pp.

The author of this little book points out in his preface that there is little hope of building a system of theology on a foundation either of philosophy or of the physical sciences, owing to the fact that neither of these branches of knowledge present results of a sufficiently authoritative character. He calls attention to the fact that the very foundations of the physical sciences themselves are at the present time open to the most serious challenge. Mathematics is the only existing body of knowledge which may be regarded as entirely authoritative. The author accordingly hopes "to throw some light, even though very dim, on some of the questions connected with our Christian beliefs." His chief aim is "to point out the remarkable agreement which exists between numerous Bible passages and some of the concepts which follow naturally from the mathematical hypothesis of higher spaces." As a popular introduction to some of the fundamental notions of four-dimensional space, the book is well done, although it seems to the reviewer that the author has rather over-emphasized the notion of reasoning by analogy. As to the value of his contribution to theology, let others speak.

J. W. Young

Ballistik. By Dr. Theodor Vahlen. Berlin, Vereinigung wissenschaftlicher Verleger, 1922. xi + 226 pp.

This book is an outgrowth of interest and investigation that were stimulated in the subject of ballistics during the World War. The author, who is professor of pure and applied mathematics in the University of Greifswald, served as an artillery officer, and during that period contributed a number of articles on ballistics to scientific and to military journals. These articles are referred to in the beginning of the book where a rather extensive bibliography on the subject is given, from the work of John Bernoulli in 1719 to the works of Cranz and of Lorenz in 1917.

The reason for writing the book is that there is no book in German literature which presents the mathematical developments of the subject in detail. Cranz has deliberately avoided these in his compendious work which is indispensable to the practical ballistician because of its useful tables and numerous illustrative examples. As no real progress in the science of ballistics is possible without a very close coordination between the theoretical and the practical, the author aims to supply a definite demand by presenting the subject from the standpoint of the mathematician.

There are four divisions of the subject: exterior ballistics, the behavior of the projectile after the powder gas has exerted its entire accelerating effect; interior ballistics, the behavior of the projectile until it begins to leave the muzzle; transition (Übergangs) ballistics which embraces the interval between exterior and interior; and, *Endballistik* which includes the functioning of shells, recoil, etc. The subject is divided further into old ballistics which rests upon the assumptions that the earth is flat, motionless, that gravity, both in magnitude and direction, and the density of the air (upon which the resistance of the air depends) are constant throughout the trajectory, and modern ballistics in which greater ranges and altitudes are encountered and in which the former assumptions are no longer adequate. The claim is made that both transition ballistics and, what I have called *modern* ballistics are treated systematically for the first time in this book. The author seems disposed to name the latter *universal* (that is, as applying to the universe instead of to the earth only) ballistics but refrains from doing so in that sense because the motion of a meteor belongs to the field of celestial mechanics and cannot be regarded a problem of what he calls "kosmische" ballistics.

Under the head of modern ballistics he gives just about what has been developed in this country during the same period, both in regard to the trajectory itself and in regard to small corrections. If there is any difference, his mathematics is more elementary. The coordinates of a point on the trajectory are expressed as power series in t (the time); in this country the coordinates of a point on the trajectory are expressed as functions of velocity, acceleration, and a quantity E which depends upon velocity, altitude, and the ballistic coefficient. The two methods are theoretically the same if we admit that our x and y may be expressed as power series in t. If different results are obtained by the two methods, it would be due to inaccuracies in tabulated physical data.

As the author has so strongly stressed the necessity of developing the mathematical and the empirical aspects of the science side-by-side as a condition precedent to making any permanent progress, he is open to criticism on exactly those grounds in his treatment of correcting for curvature of the earth. To show that this correction is of practical value he takes the extraordinarily long range of 112.8 km. (about 70 miles) and points out that in this case the correction would be 1 km. "Surely such a quantity is not to be disregarded." But does it not seem so very large only because we have so little definite knowledge in regard to the probable error of the gun fired at such a range? Would it not have been more interesting to the average reader to know how the correction for a range of 8 or 10 miles compares with the mean deviation of a series of rounds fired at these more usual ranges? Also it might be well to include a numerical example taking the gun and the target at different heights above sea-level, especially since this could be done by using the formula which is given. The same sort of criticism might be made in regard to the corrections for rain and for an increase in the temperature of the gun due to firing. Experience in this country would seem to indicate that rain would have to be further classified into mist, heavy rain, etc.

In interior ballistics the author follows Sarru and Charbonnier. The three principal formulas are as follows: (I) connecting pressure and density of loading; (II) connecting the linear velocity of the burning of the powder with density and pressure; (III) the work done in the adiabatic expansion of a gas from a finite to an infinite volume. Other familiar formulas derived in this section are those relating to variation of pressure in the barrel, the maximum pressure, and the muzzle velocity.

Although the author emphasizes the importance of transition ballistics and the fact that he presents the subject systematically for the first time, he adds nothing to the subject which is not already very familiar to those interested in the subject. The projectile does not attain its maximum velocity at the muzzle but it is accelerated by the powder gas for some little time afterward. It is also during this interval that the projectile is given a yaw the effect of which, both upon range and upon deflection, is much greater than was formerly supposed. This is one of the problems that remains to be solved and it constitutes one of the most interesting and most advanced fields of investigation in ballistics. The only contribution is a definite name for this important division of the subject.

The later chapters contain a mathematical discussion of probability as related to problems in artillery and suggestions regarding indirect fire.

The author has been successful in making his book mathematical, which was one of his aims. In the reviewer's opinion he has not developed the subject as if the mathematical problems were suggested by observations made in practical experimentation. He pointed out the necessity for this and lamented the fact that it had not been done consistently in the past. The book will be of considerable interest to the mathematician who wishes to acquaint himself with the subject of ballistics; it will be of little value to the practical artillery officer or to the investigator in practical ballistics.

J. E. Rowe

NOTES

During the Summer Meeting of the Society (see this BULLETIN, vol. 29 (1923), p. 141), a joint session will be held with the Mathematical Association of America on September 6. It is expected that Professor L. E. Dickson will address this session on behalf of the Society on the topic *Algebras and their arithmetics.*

The forty-second regular session of the San Francisco Section of the Society will be held at the University of Southern California, in Los Angeles on September 17–20, 1923, in connection with the meeting of the Pacific Division of the American Association for the Advancement of Science. This announcement cancels the previously reported date and place of meeting (see this BULLETIN, vol. 29 (1923), p. 10). Abstracts should be in the hands of the Secretary of the Section, Professor B. A. Bernstein, not later than July 26.

In September, 1920, the delegates of the American Section of the International Mathematical Union extended an invitation to the Union to hold a Mathematical Congress in New York or vicinity in 1924, and the invitation was accepted at the meeting of the Union held in Strassbourg. Since that time the Section has found it not feasible to carry out in 1924 the plan originally proposed and has been obliged to withdraw its invitation. This action was taken in order that the Union should be free to consider other invitations which might be received for the Congress.

At the meeting of the Division of Physical Sciences of the National Research Council in Washington on April 22, the following officers were elected for the year beginning June, 1923: Chairman, Oswald Veblen; Secretary, W. E. Tisdale; Executive Committee, Oswald Veblen (chairman), William Duane, E. B. Frost, Theodore Lyman, W. F. G. Swann. The following persons were elected members-at-large: Bergen Davis, S. J. Barnett, Leigh Page, F. K. Richtmyer. New members representing scientific societies were announced as follows: for the American Mathematical Society, E. V. Huntington; for the American Astronomical Society, E. W. Brown; for the American Physical Society, H. M. Randall and Paul D. Foote; for the Mathematical Association of America, H. L. Rietz.

The first award of the Bôcher Prize of this Society (see this BULLETIN, vol. 28, pp. 42–44 and p. 378) will be made at the time of the annual meeting of the Society in New York, December 27 and 28, 1923.

The members of this Society will be glad to hear that, thanks to the generosity of twenty persons, the Expense Fund estimated necessary to cover the cost of raising the proposed endowment has been completely subscribed. It is believed that this will more than take care of all overhead charges, and every cent contributed in future will go directly to the purposes for which the endowment is planned.

The ANNALI DI MATEMATICA will begin its fourth series with the next volume, the first number of which is now in press. It will be published by Nicolo Zanichelli, at Bologna, under the present board of editors.

The price for subscribers outside of Italy has been fixed at 50 French francs per volume, but subscriptions received prior to March 1, 1923, at the former rate of 40 lire, will be honored for the first volume. Owing to the generous response to the recent appeal, no further cash contributions are now needed, but more subscriptions are desired and invited. These should be sent either to the publisher in Bologna, or to Professor Virgil Snyder, Ithaca, New York.

A continuation of Professor L. P. Eisenhart's *Differential Geometry* has just appeared under the title *Transformation of Surfaces*. This is the first book to be published with the cooperation of the National Research Council in connection with its new plan of a revolving book fund for the publication of advanced treatises in mathematics. Professor Eisenhart has given his book to help in this plan, in the sense that all profits from the sale of the book go to increase the fund. The Princeton University Press, which is acting technically as the publisher of the book, has generously offered to exercise its functions without charge. The book contains 380 pages, and its price is four dollars. It is to the advantage of the fund to have purchases made directly through the Princeton University Press.

The following advanced courses in mathematics are announced at the University of Chicago for the Summer Quarter, 1923 (June 18 to August 31). All courses meet four times a week. By Professor L. E. Dickson: Differential equations from the standpoint of Lie; Arithmetic and algebra of hypercomplex numbers.—By Professor F. R. Moulton: Analytic mechanics; Theory of functions of a complex variable.—By Professor A. C. Lunn: Theory of relativity; Theory of sound.—By Professor L. S. Mordell: Theory of definite integrals; Analytic theory of numbers.—By Professor E. J. Wilczynski: Theory of linear differential equations.—By Professor J. W. A. Young: Selected topics in mathematics.—By Professor E. J. Moulton: Advanced calculus.—By Dr. Mayme I. Logsdon: Higher plane curves.—By Professor A. F. Carpenter: Solid analytic geometry.

Mr. Dan Brouse, a graduate of Purdue University, has been appointed assistant professor of mathematics at Baker University, Baldwin, Kansas.

At Kansas State Agricultural College, Mr. W. H. Rowe, Mr. W. C. Janes, and Miss Thirza A. Mossman have been appointed instructors in mathematics.

At Iowa State College, Messrs. F. A. Brandner, H. P. Doole, P. O. Robinson, and Henry Giese have been appointed instructors in mathematics.

Mr. Macon Reed, of Columbia University, has been appointed head of the department of mathematics at Hampden-Sidney College.

Miss Rose B. Wood, a graduate of Barnard College, has been appointed professor of mathematics at Greenville (S. C.) Woman's College.

At the University of South Carolina, Mr. M. A. Hill and Mr. C. C. Edwards have been appointed instructors in mathematics.

Mr. R. G. Demaree, of the University of Chicago, has been appointed head of the department of mathematics and physics at Kentucky Wesleyan College.

NEW PUBLICATIONS

I. PURE MATHEMATICS

APPELL (P.) et LACOUR (E.). Principes de la théorie des fonctions elliptiques et applications. 2e édition. Paris, Gauthier-Villars, 1922.

BAKER (H. F.). Principles of geometry. Volume 2: Plane geometry, conics, circles, non-euclidean geometry. Cambridge, University Press, 1922. 16 + 244 pp.

BELTRAMI (E.). Opere matematiche. Tomo 4 ed ultimo. Milano, Hoepli, 1920. 4to. 554 pp.

BRADLEY (F. H.). The principles of logic. 2d edition. Volumes 1–2. London, Oxford University Press, 1922. 28 + 388 + 350 pp.

FUBINI (G.). Lezioni di analisi matematica. 4a edizione, interamente rifusa. Torino, Società Tipografico-Editrice Nazionale, 1920. 8 + 470 pp.

GOURSAT (E.). Cours d'analyse mathématique. 3e édition. Tome 3. Paris, Gauthier-Villars, 1923.

HAAS (A.). Vektoranalysis in ihren Grundzügen und wichtigsten physikalischen Anwendungen. Berlin, Vereinigung wissenschaftlicher Verleger, 1922. 6 + 149 pp.

KRAÏTCHIK (M.). Décomposition de $a^n \pm b^n$ en facteurs dans le cas où nab est un carré parfait avec une table des décompositions numériques pour toutes les valeurs de a et b inférieures à 100. Paris, Gauthier-Villars, 1922.

KUMM (E.). Das Potential gewisser homogener Rotationsovaloide. (Diss., Halle-Wittenberg.) Halle, Buchdruckerei Hohmann, 1922. 45 pp.

LACOUR (E.). See APPELL (P.).

LEHNEN (M.). Eine Theorie der Raumkurven 3. 0. auf Grundlage der Invariantentheorie. (Diss., Bonn.) Bonn, Druck von T. Wurm, 1921. 48 pp.

MUICA (I.). Quelques observations élémentaires sur les nombres entiers. Théorème de Fermat. 2e édition. Bucarest, Viata Românească, 1922. 12 pp.

NEVILLE (E. H.). Prolegomena to analytical geometry in anisotropic euclidean space of three dimensions. Cambridge, University Press, 1922. 22 + 368 pp.

POPPOVICH (N. M.). Die Lehre vom diskreten Raum in der neueren Philosophie. Wien und Leipzig, Braumüller, 1922. 2 + 89 pp.

SEN GUPTA (J.). The fundamentals. Volume 1. Calcutta, Kar Majumder and Company, 1921. 8vo. 85 pp.

WEISNER (L.). Groups whose maximal cyclic subgroups are independent. (Diss., Columbia.) New York, 1923. 17 pp.

WIELEITNER (H.). Geschichte der Mathematik. I: Von ältesten Zeiten bis zur Wende des 17tes Jahrhunderts. (Sammlung Göschen.) Berlin, Vereinigung wissenschaftlicher Verleger, 1922. 136 pp.

II. APPLIED MATHEMATICS

ADLER (F.). Ortzeit, Systemzeit, Zonenzeit und das ausgezeichnete Bezugssystem der Elektrodynamik. Eine Untersuchung über die Lorentzsche und die Einsteinsche Kinematik. Wien, Wiener Volksbuchhandlung Ignaz Brand, 1920. 8vo. 16 + 237 pp.

BHAGWAT (V. K.). Calculations in organic chemistry. Bombay, S. Govind and Company, 1921. 11 + 138 pp.

BJERKNES (V.). Untersuchungen über elektrische Resonanz. Leipzig, Barth, 1922. 32 + 129 pp.

BOHR (N.). The theory of spectra and atomic constitution. Three essays. Cambridge, University Press, 1922. 10 + 126 pp.

BOUASSE (H.). Hydrostatique. Manomètres, baromètres, pompes. Equilibre des corps flottants. Paris, Delagrave, 1923. 24 + 480 pp.

BRIDGMAN (P. W.). Dimensional analysis. New Haven, Yale University Press, 1922. 5 + 112 pp.

CHAPMAN (R. W.). The elements of astronomy for surveyors. 2d edition, revised. London, Griffin, 1922. 261 pp.

DROSNE (P.). Le beau, le mauvais temps et la relativité. Paris, Doin, 1922. 18mo. 68 pp.

EUCKEN (A.). Grundriss der physikalischen Chemie. Leipzig, Akademische Verlagsgesellschaft, 1922. 8 + 492 pp.

FISCHER (L.). Die Wärme ein Gas! Eine neue Theorie der Wärme und der übrigen feinen Stoffe. Leipzig, H. A. Ludwig Degener, 1922. 61 pp.

GRÖBER (H.). Die Grundgesetze der Wärmeleitung und des Wärmeüberganges. Berlin, Springer, 1921. 271 pp.

HANOCQ (C.). Les pompes centrifuges à haute pression. Paris et Liège, Béranger, 1921. 8vo. 160 pp.

INSTITUT international de Physique Solvay. Atomes et électrons. Rapports et discussions du Conseil de Physique tenu à Bruxelles du 1er au 6 avril 1921. Paris, Gauthier-Villars, 1921. 8 + 273 pp.

KRIEMLER (C.). Technische Mechanik. Ein Lehrbuch der Statik und Dynamik starrer und nachgiebiger Körper. 2te, vermehrte und verbesserte Auflage. Stuttgart, 1920.

LEBERT (E.). Electricité, magnétisme, gravitation, mécanique, théorie de la chaleur. Vannes, Lafolye frères, 1922.

LORING (F. H.). Definition of the æther. London, H. O. Lloyd, 1922. 16 pp.

MÜLLER (A.). Der Gegenstand der Mathematik mit besonderer Beziehung auf die Relativitätstheorie. Braunschweig, Vieweg, 1922. 8 + 94 pp.

MÜLLER (E.). Technische Uebungsaufgaben für darstellende Geometrie. Heft 5. Leipzig und Wien, Deuticke, 1921.

NEWEST (T.). Gegen Einstein: Die Erfahrung im Weltall. Einige Weltprobleme. Allgemeinverständliche Abhandlungen. 2te Auflage. Wien, Frisch, 1921. 8vo. 95 pp.

PECHEUX (H.). Traité d'électricité industrielle. Paris, Delagrave, 1922. 2 volumes. 834 + 376 pp.

PETROVITCH (M.). Mécanismes communes aux phénomènes disparates. Paris, Alcan, 1921. 8vo. 279 pp.

PFLÜGER (A.). Das Einsteinsche Relativitätsprinzip gemeinverständlich dargestellt. 3te Auflage. Bonn, 1920.

PODEVYN (P. A.). Traité de mécanique, à l'usage des ouvriers, dessinateurs, et techniciens-mécaniciens. Paris, Librairie Desforges, 1922. 16mo. 169 pp.

RINNE (F.). Das feinbauliche Wesen der Materie nach dem Vorbilde der Kristalle. 3te Auflage. Berlin, Gebrüder Borntraeger, 1922. 8 + 168 pp.

RIPKE-KÜHN (L.). Kant contra Einstein. Erfurt, Keyserschen Buchhandlung, 1920.

SCHAU (A.). Festigkeitslehre. 2te Auflage. (Aus Natur und Geisteswelt.) Leipzig, Teubner, 1921. 111 pp.

SEPP (H.). Stoff und Kraft in Verbindung mit Raum und Zeit, nach den neuesten Forschungen über Atomzerfall sowie nach der Quanten- und der Einstein'schen Relativitätstheorie. Berlin-Steglitz, Heimatverlag M. Hiemesch, 1921. 8vo. 32 pp.

SOLVAY. See INSTITUT.

TUMLIRZ (O.). Die Zustandsausgleichung des flüssigen und dampfförmigen Wassers. Wien, Hölder, 1921. 8vo.

VERSCHOYLE (W. D.). The evolution of atoms and isotopes. London, Keliher, 1922. 40 pp.

WANTOCH (R.). Die Einsteinsche Relativitätstheorie. Kurze, für jedermann verständliche Besprechung. Wien, Hölder, 1921. 8vo. 15 pp.

WEYL (H.). Raum, Zeit, Materie. Vorlesungen über allgemeine Relativitätstheorie. 5te, umgearbeitete Auflage. Berlin, Springer, 1923. 8 + 338 pp.

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

THE SUMMER MEETING OF THE SOCIETY, at VASSAR COLLEGE, September 6–8, 1923.

Abstracts must be in the hands of the Secretary of the Society not later than August 8.

THE SAN FRANCISCO SECTION, at LOS ANGELES, September 17–20, 1923.

Abstracts must be in the hands of the Secretary of the Section, B. A. Bernstein, not later than July 26.

NEW YORK CITY, October 27, 1923.

Abstracts must be in the hands of the Secretary of the Society not later than October 13.

R. G. D. RICHARDSON,
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Any communication intended for this Bulletin may be addressed to the American Mathematical Society, Prince and Lemon Streets, Lancaster, Pa., or may be sent to separate officials of the Society as follows:

Articles for insertion in the Bulletin should be addressed to E. R. Hedrick, Editor of the Bulletin, 304 Hicks Ave., Columbia, Mo. Reviews should be sent to J. W. Young, Dartmouth College, Hanover, N. H. Notes should be sent to R. W. Burgess, Brown University, Providence, R. I.

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